

The University of Chicago
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CONCERNING CERTAIN ELLIPTIC MODU- LAR FUNCTIONS OF SQUARE RANK

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

(DEPARTMENT OF MATHEMATICS)

BY
JOHN ANTHONY MILLER

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AUTOBIOGRAPHY

I was born near Greensburg, Indiana, on the sixteenth of December, 1859. I received my early education in the public schools of Indiana. I entered Indiana University in 1886, and received from this University the degree of Bachelor of Arts in 1890. I received the degree, Master of Arts from the Leland Stanford Junior University in 1893. I was a graduate student in the University of Chicago two years, 1895-97. My studies in mathematics were directed by Professors Moore, Bolza and Maschke; in astronomy by Doctors See and Laves. I gratefully acknowledge my indebtedness to these men for encouragement and assistance given me while I was pursuing my graduate studies.

JOHN ANTHONY MILLER

Concerning Certain Elliptic Modular Functions of Square Rank.

BY JOHN A. MILLER.

INTRODUCTION.

In his "Vorlesungen*" über die Theorie der Elliptischen Modulfunctionen," Professor Felix Klein has defined the functions†

$$X_{\frac{\alpha}{n}}(u|\omega_1, \omega_2) = C_{\alpha} \prod_{\mu=0}^{n-1} \sigma_{\frac{\alpha}{n} + \epsilon, \frac{\mu}{n}}(u|\omega_1, \omega_2), \quad (1)$$

where u is the fundamental variable of the elliptic function; ω_1, ω_2 , are the periods of the elliptic integral of the first kind; $\epsilon = 0$ or $\frac{1}{2}$ according as n is odd or even, C_{α} is a quantity independent of u , and

$$\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(u|\omega_1, \omega_2) = e^{\frac{\lambda\eta_1 + \mu\eta_2}{n} \left(u - \frac{\lambda\omega_1 + \mu\omega_2}{2n}\right)} \sigma\left(u - \frac{\lambda\omega_1 + \mu\omega_2}{n} \middle| \omega_1, \omega_2\right), \quad (2)$$

where $\sigma(u|\omega_1, \omega_2)$ is the ordinary Weierstrassian sigma-function; η_1, η_2 , are the periods of elliptic integrals of the second kind and α, n, λ, μ are integers. He has shown that these functions are of rank (stufe‡) n .

I have, in what follows, investigated certain properties of these functions, and functions derived from them, when n is a square number, and have treated in detail the particular cases where $n = 9$ and $n = 4$.

In the future, when there is no possible ambiguity, I shall write, instead of $X_{\frac{\alpha}{n}}(u|\omega_1, \omega_2)$, the briefer forms $X_{\alpha}(u|\omega_1, \omega_2)$ or $X_{\alpha}(u)$; and instead of

* Hereafter I shall refer to this treatise as "Klein-Fricke."

† Klein-Fricke, Vol. II, p. 261.

‡ Klein, "Ueber die Normal Curven der n^{ter} Ordnung."

$\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(u | \omega_1, \omega_2)$ the briefer forms $\sigma_{\lambda, \mu}(u)$ and $\sigma_{\lambda, \mu}(0) = \sigma_{\lambda, \mu}$. I shall enumerate without proof certain properties of the functions defined by (1), viz.

$$X_{a+n}(u) = X_a(u), \quad (3)^*$$

$$X_a(-u) = (-1)^n X_{n-a}(u), \quad (4)^*$$

$$X_a\left(u + \frac{\lambda\omega_1 + \mu\omega_2}{n}\right) = (-1)^{n(\lambda + \mu)} e^{(\lambda\eta_1 + \mu\eta_2)\left(u + \frac{\lambda\omega_1 + \mu\omega_2}{2n}\right)} e^{\frac{\mu\pi i(\lambda - 2a)}{n}} X_{a-\lambda}(u), \quad (5)^*$$

$$X_a(u) = (-1)^{na} e^{a\eta_1\left(u - \frac{\alpha\omega_1}{2n}\right)} X_0\left(u - \frac{\alpha\omega_1}{n}\right). \quad (6)^*$$

There are n distinct functions $X_a(u)$, and n only, and they are linearly independent.

There are two cases to be considered: Case I, $n \equiv 1 \pmod{2}$; Case II, $n \equiv 0 \pmod{2}$.

Case I.

1.—*Transformation of the $X(u)$ by the Modular Substitutions when n is Odd.*

The arbitrary constant C_a in $X_a(u)$ may be so chosen that we may write, in case n is odd,

$$X_a(u | \omega_1, \omega_2) = (-1)^a \sqrt[n]{\frac{\bar{\Delta}}{\Delta^n}} e^{-G_1 u^2} \sigma_{\frac{a}{n}, 0}\left(u | \omega_1, \frac{\omega_2}{n}\right), \quad (7)^\dagger$$

or

$$X_a(u | \omega_1, \omega_2) = (-1)^a - i \frac{e^{\frac{u^2 n \eta_1}{2\omega_1}}}{\sqrt[n]{\Delta^n}} \cdot \sqrt{\frac{2\pi i}{\omega_1}} \theta_1\left(-\frac{\pi}{\omega_1} \frac{(u - \alpha\omega_1)}{n}, e^{-2\frac{\pi i \omega_2}{n \omega_1}}\right), \quad (8)$$

‡ where $\Delta = \Delta\left(\omega_1, \frac{\omega_2}{n}\right)$ and

$$G_1 = \frac{\bar{\eta}_1 - n\eta_1}{2\omega_1} = \frac{\bar{\eta}_2 - n\eta_2}{2\omega_2},$$

where, owing to the cogredience $||$ of ω_1, ω_2 and η_1, η_2 ,

$$\bar{\eta}_1 = \eta_1 \quad \text{and} \quad \bar{\eta}_2 = \frac{\eta_2}{n}.$$

* Klein-Fricke, Vol. II, p. 264, et seq.

† Klein-Fricke, Vol. II, p. 277.

‡ For definition of Δ , see Klein-Fricke, Vol. I, p. 15.

$||$ Klein-Fricke, Vol. I, p. 122.

Professor Klein has shown* that there exists a finite group of linear substitutions on $X_a(u)$, of order $2\mu(n)$ which may be generated by repetitions and combinations of the operators S and T , which are defined as follows:

$$\begin{aligned} S: \quad \omega'_1 &= \omega_1 + \omega_2, & T: \quad \omega'_1 &= -\omega_2, \\ \omega'_2 &= \omega_2, & \omega'_2 &= \omega_1, \end{aligned} \quad (9)$$

and that if $\varepsilon = e^{\frac{2\pi i}{n}}$,

$$\left. \begin{aligned} S: \quad X'_a(u) &= \varepsilon^{\frac{-(n-a)\alpha}{2}} X_a(u), \\ T: \quad X'_a(u) &= \frac{i^{\frac{n-1}{2}}}{\sqrt{n}} \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} X_\beta(u), \end{aligned} \right\} \quad (10)$$

Where $S: X'_a(u)$ means that $X_a(u)$ has been subjected to the operation S .

We shall now define a set of $(n-1)/2$ distinct functions† of ω_1 and ω_2 by the following equation:

$$X_a(0) = z_a(\omega_1, \omega_2) = z_a \text{ (say)}. \quad (11)$$

If we put $u=0$ and $n=9$ in equations (10), we obtain a group of quaternary substitutions on the z_a generated by

$$\left. \begin{aligned} S: \quad z'_a &= \varepsilon^{\frac{-(9-a)\alpha}{2}} z_a, \\ T: \quad z'_a &= \frac{1}{3} \sum_{\beta=0}^8 \varepsilon^{-\alpha\beta} z_\beta \cdot (\alpha = 1 \dots 4). \end{aligned} \right\} \quad (12)$$

The order of this group is $(9)\phi(9)\psi(9) = 648$. We shall call it $\bar{G}_{648}$.

2.—The Biquadratic Relations of the z_a .

If we substitute in the well-known σ -relation,‡

$$\begin{aligned} &\sigma(u_1 + u_2) \sigma(u_3 + u_4) \sigma(u_1 - u_2) \sigma(u_3 - u_4) \\ &+ \sigma(u_1 + u_3) \sigma(u_1 - u_3) \sigma(u_4 + u_2) \sigma(u_4 - u_2) \\ &+ \sigma(u_1 + u_4) \sigma(u_1 - u_4) \sigma(u_2 + u_3) \sigma(u_2 - u_3) = 0, \end{aligned}$$

* Klein-Fricke, Vol. II, p. 296.

† To show that there actually exists a set of functions of ω_1, ω_2 such as we have defined, see Klein-Fricke, Vol. II, p. 267, equation (12).

‡ Schwarz, Formeln und Lehrsätze zum Gebrauche der Elliptischen Functionen, p. 47.

$u_i = \frac{u}{2} - \frac{\alpha_i \omega_1}{n}$, we obtain, after reduction,

$$X_{\alpha_1 + \alpha_2} X_{\alpha_3 + \alpha_4} z_{\alpha_2 - \alpha_1} z_{\alpha_4 - \alpha_3} + X_{\alpha_1 + \alpha_3} X_{\alpha_2 + \alpha_4} z_{\alpha_3 - \alpha_1} z_{\alpha_2 - \alpha_4} + X_{\alpha_1 + \alpha_4} X_{\alpha_2 + \alpha_3} z_{\alpha_4 - \alpha_1} z_{\alpha_3 - \alpha_2} = 0. \quad (a)^*$$

Put in this equation $u = 0$, and we obtain

$$z_{\alpha_1 + \alpha_2} z_{\alpha_3 + \alpha_4} z_{\alpha_2 - \alpha_1} z_{\alpha_4 - \alpha_3} + z_{\alpha_1 + \alpha_3} z_{\alpha_4 + \alpha_2} z_{\alpha_3 - \alpha_1} z_{\alpha_2 - \alpha_4} + z_{\alpha_1 + \alpha_4} z_{\alpha_2 + \alpha_3} z_{\alpha_4 - \alpha_1} z_{\alpha_3 - \alpha_2} = 0. \quad (b)$$

The left side of equation (b) vanishes for all values of ω_1, ω_2 . It is our purpose to find all such expressions for the case $n = 9$. Inspection shows that the left side of equation (b) vanishes identically in z_a , if any subscript of z_a is congruent zero modulo n .

If we write $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = s$, then any pair of X 's occurring in (a) may be put in the form $X_i \cdot X_{s-i}$. Therefore, by virtue of equation (3), s runs through the residue system, modulo 9.

By virtue of equation (a), any X_a -pair $X_i X_{s-i}$ can be expressed in terms of two pairs, $X_k X_{s-k} X_l X_{s-l}$, arbitrarily chosen; and any other relation can be obtained by the elimination of one of these pairs from two independent equations, provided the determinant of the coefficients of $X_k X_{s-k}$ and $X_l X_{s-l}$ does not vanish.

Choose then: the pairs $X_1 X_{s-1}, X_2 X_{s-2}$; whence

$$\left. \begin{aligned} \alpha_1 + \alpha_3 &\equiv 1, \\ \alpha_2 + \alpha_4 &\equiv s - 1, \\ \alpha_1 + \alpha_4 &\equiv 2, \\ \alpha_3 + \alpha_2 &\equiv s - 2, \end{aligned} \right\} \pmod{9}.$$

Making substitutions in equation (b) consistent with the foregoing, we obtain all relations except in the case $s = 3$, when the equation vanishes identically in z_a . They are

$$\begin{aligned} z_2^3 z_1 - z_3^3 z_2 + z_4^3 z_1 &= 0 = S_1 \text{ (say),} \\ z_1^3 z_4 + z_2^3 z_4 - z_3^3 z_1 &= 0 = S_2 \text{ (say),} \\ -z_1^2 z_2 + z_2^2 z_4 + z_1 z_4^2 &= 0 = S_3 \text{ (say).} \end{aligned}$$

* Here, and throughout this section, X_a means $X_a(u)$.

Since no relation exists in case $s = 3$ involving the X_a -pair X_1X_2 , there remains only one relation, and it is found to be

$$-z_2z_1^3 + z_3^3z_4 + z_2z_4^3 = 0 = S_4 \text{ (say).}$$

If we regard z_1, z_2, z_3, z_4 as homogeneous coordinates in a space of three dimensions, and if $S_1 - z_4S_4 = z_2S_5$, then the curve

$$S_4 = 0, \quad S_5 = 0$$

is the complete intersection of the four surfaces

$$S_1 = 0 = S_2 = S_3 = S_4.$$

The curve is of ninth degree.

3.—The Invariant Tetrahedron. The Functions t_a .

Regarding again z_1, z_2, z_3, z_4 as homogeneous coordinates in a space of three dimensions, we shall show first, that no plane is invariant under the substitutions of $\bar{G}_{648}$. Let

$$A_1z_1 + A_2z_2 + A_3z_3 + A_4z_4 = 0$$

be the equation of any plane. Then

$$*S: A_1z_1 + A_2z_2 + A_3z_3 + A_4z_4 = \epsilon^5 A_1z_1 + \epsilon^2 A_2z_2 + A_3z_3 + \epsilon^8 A_4z_4.$$

Hence the only planes unchanged by S are $z_1 = 0, z_2 = 0, z_3 = 0, z_4 = 0$. None of these are unchanged by T . This proves the theorem.

Let us find now the equations of all tetrahedrons invariant under the group of substitutions $\bar{G}_{648}$.

Any tetrahedron invariant under S is evidently invariant under S^6 . Let

$$A_1z_1 + A_2z_2 + A_3z_3 + A_4z_4 = 0 = P$$

be the equation of a plane of the invariant tetrahedron.

$$S^3: A_1z'_1 + A_2z'_2 + A_3z'_3 + A_4z'_4 = A_1\epsilon^6z_1 + A_2\epsilon^6z_2 + A_3z_3 + A_4\epsilon^6z_4 = Q = 0,$$

$$S^6: A_1z'_1 + A_2z'_2 + A_3z'_3 + A_4z'_4 = A_1\epsilon^3z_1 + A_2\epsilon^3z_2 + A_3z_3 + A_4\epsilon^3z_4 = R = 0.$$

* By this notation is meant that we apply the operation S to ΣA_iz_i and derive $\Sigma A'_iz'_i$. But we know that this operation yields the expression on the right of the equation. We shall hereafter use this notation.

Since $(S^3)^3 = 1$,

the operation S^3 either leaves the planes $P=0$, $Q=0$ and $R=0$ each unchanged or permutes them. Hence S leaves at least one plane of the tetrahedron unchanged.

Suppose $P_1 = A_1z_1 + A_2z_2 + A_3z_3 + A_4z_4 = 0$

be the equation of this plane, it is invariant under S if, and only if,

$$\left. \begin{array}{l} A_1 = A_2 = A_4 = 0 \\ A_3 = 0. \end{array} \right\} \quad (\text{A})$$

Let us choose first, the first set of relations; then

$$z_3 = 0$$

is one of the planes of the tetrahedron. The equation of the other three planes may be found, if an invariant tetrahedron exists, by subjecting z_3 to any three operations of the group such as $T^\nu S^\mu$, $\nu \neq 0$. The equation of the tetrahedron is found to be

$$z_3(z_1 - z_2 + z_4)(\epsilon^3 z_4 - z_2 + \epsilon^6 z_4)(z_1 - \epsilon^3 z_2 + \epsilon^6 z_4) = 0. \quad (13)$$

The plane $A_1z_1 + A_2z_2 + A_4z_4 = 0 = L$ (say),

is unchanged by S^3 , and hence, reasoning as before, one plane at least of the tetrahedron is unchanged by S . The only planes unchanged by S are

$$z_1 = 0; \quad z_2 = 0, \quad z_4 = 0 \quad \text{and} \quad z_3 = 0.$$

The last one we have discussed. Let us examine the others: Let

$$\begin{aligned} T &: z_1 = L, \\ TS &: z_1 = M, \\ TS^2 &: z_1 = N. \end{aligned}$$

Since $S: N$ produces neither L , M nor z_1 , the tetrahedron

$$z_1 L M N = 0$$

is not invariant under the substitutions of the group. Hence $z_1 = 0$ is not one of the planes of the tetrahedron. In a similar way, it may be shown that $z_2 = 0$ and $z_4 = 0$ are not planes of the tetrahedron. Hence, the only tetrahe-

dron invariant under $\bar{G}_{648}$ is given by equation (13). Put now

$$\left. \begin{aligned} 3t_1 &= z_1 - z_2 + z_4, \\ 3t_2 &= \varepsilon^5 z_1 - \varepsilon^2 z_2 + \varepsilon^8 z_4, \\ 3t_3 &= \varepsilon z_1 - \varepsilon^4 z_2 + \varepsilon^7 z_4, \\ 3t_4 &= (\varepsilon^3 - \varepsilon^6) z_3. \end{aligned} \right\} \quad (14)$$

We have immediately

$$S: \left. \begin{aligned} t'_1 &= t_2, \\ t'_2 &= t_3, \\ t'_3 &= \varepsilon^6 t_1, \\ t'_4 &= t_4, \end{aligned} \right\} \quad (15) \quad T: \left. \begin{aligned} t'_1 &= -t_4, \\ t'_2 &= \varepsilon^3 t_3, \\ t'_3 &= -\varepsilon^6 t_2, \\ t'_4 &= t_1. \end{aligned} \right\} \quad (16)$$

The substitutions defined by (15) and (16) form a group of unary substitutions, holodric isomorphic with $\bar{G}_{648}$. I shall call this group G_{648} .

The functions $\sigma_{\frac{\lambda}{3}, \frac{\mu}{3}}$ when subjected to S and T defined by (9) are interchanged among themselves as follows:

$$S: \begin{aligned} \sigma'_{0,1} &= \sigma_{0,1}, \\ \sigma'_{1,0} &= \sigma_{1,1}, \\ \sigma'_{1,1} &= \sigma_{1,2}, \\ \sigma'_{1,2} &= \varepsilon^6 \sigma_{1,0}, \end{aligned} \quad T: \begin{aligned} \sigma'_{0,1} &= \sigma_{1,0}, \\ \sigma'_{1,0} &= -\sigma_{0,1}, \\ \sigma'_{1,1} &= \varepsilon^3 \sigma_{1,2}, \\ \sigma'_{1,2} &= -\varepsilon^6 \sigma_{1,1}. \end{aligned}$$

Comparing the substitutions on the $\sigma_{\frac{\lambda}{3}, \frac{\mu}{3}}$ with those* on $t_{\frac{a}{9}}$, we see that $\sigma_{\frac{1}{3}, \frac{2}{3}}, \sigma_{\frac{2}{3}, \frac{1}{3}}$,

$\sigma_{\frac{1}{3}, \frac{3}{3}}, \sigma_{\frac{3}{3}, \frac{1}{3}}$, are cogredient† with $t_{\frac{1}{3}}, t_{\frac{2}{3}}, t_{\frac{3}{3}}, t_{\frac{4}{3}}$.

The functions $\sigma_{\frac{\lambda}{3}, \frac{\mu}{3}}$ and $t_{\frac{a}{9}}$ are functions of ω_1, ω_2 .

We may find the relations connecting them by considering the theory of the transformation of the θ -functions in terms of which both may be expressed. They are:

$$\left. \begin{aligned} \sigma_{\frac{1}{3}, \frac{1}{3}} &= \Delta t_{\frac{1}{3}}, \\ \sigma_{\frac{1}{3}, \frac{2}{3}} &= \Delta t_{\frac{2}{3}}, \\ \sigma_{\frac{1}{3}, \frac{3}{3}} &= \Delta t_{\frac{3}{3}}, \\ \sigma_{\frac{1}{3}, \frac{4}{3}} &= \Delta t_{\frac{4}{3}}. \end{aligned} \right\} \quad (\alpha)$$

* Since t_a are derived from z_a and z_a from $X_a(u)$, when it is especially desirable to emphasize the relation of a to 9, we shall write $t_{\frac{a}{9}}$ instead of t_a and $z_{\frac{a}{9}}$ instead of z_a .

† Burnside and Panton, Theory of Equations, p. 384.

4.—*The Invariant Functions of t_a .*

By an invariant function we shall mean a rational integral function of t_a , defined by equation (14), which does not vanish identically in t_a , with degree greater than zero, and which remains unchanged by all the substitutions of G_{648} . Such functions exist, for the elementary symmetric functions of $t_1^6, t_2^6, t_3^6, t_4^6$ are invariant under this group.

An i -lettered term is one containing i letters.

An i -lettered function is a rational integral function containing only i -lettered terms.

A form is a homogeneous integral function. We shall now seek all invariant functions of t_a .

Any invariant integral function F may be written $F = \sum_{\kappa=0}^4 a_{\kappa} F_{\kappa}$, where F_{κ} are i -lettered functions and a_{κ} are independent of t_a and do not all vanish. Since, under any substitution of G_{648} , we change neither the number of letters nor the degree of any term, it follows that each F_{κ} consists of the sum of i -lettered forms, and is, therefore, itself an invariant form. Any four-lettered form is a rational integral function of $t_1 t_2 t_3 t_4$ and one-, two- and three-lettered forms.

We shall now derive all three-lettered invariant forms. Choose arbitrarily one of the terms R_i , where

$$R_1 = t_1^{\alpha} t_2^{\beta} t_3^{\gamma}, \quad R_2 = t_1^{\alpha} t_2^{\beta} t_4^{\gamma}, \quad R_3 = t_1^{\alpha} t_3^{\beta} t_4^{\gamma}, \quad R_4 = t_2^{\alpha} t_3^{\beta} t_4^{\gamma}, \quad (a)$$

where α, β, γ are integers. Call the term so chosen the originating term and denote it by R . Let the terms derived by subjecting R to the substitutions of G_{648} be denoted by R_i . It is evident that R_1, R_2, R_3 and R_4 will be of their number. Let $F_{\alpha, \beta, \gamma}$ be an invariant three-lettered function of degree $\alpha + \beta + \gamma$. Then $F_{\alpha, \beta, \gamma}$ is the simplest symmetric function of R_i ; for, since it is unchanged by S and T , it is unchanged by $S^{\mu} T^{\nu}$. Therefore, the R_i are all found among the terms of $F_{\alpha, \beta, \gamma}$. Moreover, $F_{\alpha, \beta, \gamma}$ is of the form

$$F_{\alpha, \beta, \gamma} = \sum a_i R_i,$$

where a_i is independent of t_a . For, $F_{\alpha, \beta, \gamma}$ is a form, and any operation except the addition of terms such as $a_i R_i$ either destroys its homogeneity or changes its

degree. Some substitution σ of G_{648} exists that changes R_i into R_j , and the necessary and sufficient condition that $\sigma : F'_{\alpha, \beta, \gamma} = F_{\alpha, \beta, \gamma}$ is that $\alpha_i = \alpha_j$.

Now, if R_i be either of the terms (a), then

$$T^2 : R_i = (-1)^{(\alpha + \beta + \gamma)} R_i.$$

Hence, when we form the simplest symmetric function of the originating term and those obtained by operating with T^2 , this function vanishes identically unless

$$\alpha + \beta + \gamma \equiv 0 \pmod{2}. \quad (b)$$

Hence, we conclude that this congruence holds.

Also, if

$$\begin{aligned} R &= t_1^\alpha t_2^\beta t_3^\gamma, \\ S^3 : R' &= \epsilon^{6(\alpha + \beta + \gamma)} R, \\ S^6 : R' &= \epsilon^{3(\alpha + \beta + \gamma)} R. \end{aligned}$$

Hence, the simplest symmetric function formed from the originating term and the terms resulting from the repeated applications of S^3 , vanishes identically unless

$$\alpha + \beta + \gamma \equiv 0 \pmod{3}. \quad (c)$$

Hence, we conclude that congruence (c) holds; or that no three-lettered invariant function exists. But we have shown that some do exist.

If we choose R_3 as the originating term, we may show in a similar manner that

$$\alpha + \beta \equiv 0 \pmod{3},$$

therefore,

$$\gamma \equiv 0 \pmod{3}.$$

In a similar way we can show that

$$\alpha \equiv 0 \pmod{3}$$

and

$$\beta \equiv 0 \pmod{3}.$$

We conclude, therefore, that in all three-lettered forms the exponents satisfy the congruences

$$\left. \begin{aligned} \alpha + \beta + \gamma &\equiv 0 \pmod{6}, \\ \alpha &\equiv \beta \equiv \gamma \equiv 0 \pmod{3}. \end{aligned} \right\} \quad (d)$$

Choosing as our originating term $t_1^\alpha t_2^\beta t_3^\gamma$, we obtain as a three-lettered inva-

riant form

$$F_{\alpha, \beta, \gamma} = \left. \begin{aligned} & t_1^\alpha t_2^\beta t_3^\gamma + t_2^\alpha t_3^\beta t_1^\gamma + t_3^\alpha t_1^\beta t_2^\gamma \\ & + (-1)^\alpha [t_1^\alpha t_3^\beta t_4^\gamma + t_2^\alpha t_1^\beta t_4^\gamma + t_3^\alpha t_2^\beta t_4^\gamma] \\ & + (-1)^\beta [t_4^\alpha t_1^\beta t_3^\gamma + t_4^\alpha t_2^\beta t_1^\gamma + t_4^\alpha t_3^\beta t_2^\gamma] \\ & + (-1)^\gamma [t_3^\alpha t_4^\beta t_1^\gamma + t_1^\alpha t_4^\beta t_2^\gamma + t_2^\alpha t_4^\beta t_3^\gamma]. \end{aligned} \right\} \quad (17)$$

Giving to α, β, γ in (17) all values satisfying the congruences (d), we obtain all possible three-lettered invariant forms. For, by making suitable interchanges of α, β, γ , we can find among the terms of (17) any of the four possible originating terms.

If we put in (17) one exponent, $\gamma = 0$, we get all two-lettered invariant functions,

$$F_{\alpha, \beta} = \left. \begin{aligned} & t_1^\alpha t_2^\beta + t_2^\alpha t_3^\beta + t_3^\alpha t_1^\beta \\ & + (-1)^\alpha (t_1^\alpha t_3^\beta + t_2^\alpha t_1^\beta + t_3^\alpha t_2^\beta) \\ & + (-1)^\beta (t_4^\alpha t_1^\beta + t_4^\alpha t_2^\beta + t_4^\alpha t_3^\beta) \\ & + (t_3^\alpha t_4^\beta + t_1^\alpha t_4^\beta + t_2^\alpha t_4^\beta). \end{aligned} \right\} \quad (18)$$

Put in (18), $\beta = 0$ and $F_{\alpha, 0} = F_\alpha$, we obtain

$$F_\alpha = 3(t_1^\alpha + t_2^\alpha + t_3^\alpha + t_4^\alpha). \quad (19)$$

5.—*The Basis of the System of Forms.*

Giving to α, β, γ all possible values satisfying the congruences (d), we obtain from equations (17)–(19) an infinity of forms, all invariant under the group G_{648} . And, as we have shown that they are the only forms so constituted, we have a totality of forms definitely defined. We shall designate this totality of forms by H . By Hilbert's *Law we can express any form, F , of H , in the following form :

$$F = \sum_{i=1}^{\mu} A_i F_i, \text{ where } \mu \text{ is finite,}$$

† and A_i are forms of H . The forms F_i is called the basis of the system H .

* Hilbert, "Ueber die Theorie der Algebraischen Formen," in *Mathematische Annalen*, Band 36. Weber, *Lehrbuch der Algebra*, Band 2, p. 165.

† Weber, *Lehrbuch der Algebra*, Band 2, p. 170.

Problem : To find the basis of H .

In order to solve the problem, we need the set of identities (a) (g), the truth of which may be verified by computation. The summation under the summation signs, and the products under the product signs, are made from 1 4, although it is not so indicated in the equations of the present section.

$$\sum t_i^6 \cdot F_{\alpha, \beta, \gamma} = F_{\alpha, \beta+6, \gamma} + F_{\alpha, \beta, \gamma+6} + F_{\alpha+6, \beta, \gamma} + \Pi t_i^6 \cdot F_{\alpha-6, \beta-6, \gamma-6}, \quad (a)$$

$$\begin{aligned} \sum t_i^6 t_k^6 \cdot F_{\alpha, \beta, \gamma} = & F_{\alpha, \beta+6, \gamma+6} + F_{\alpha+6, \beta, \gamma+6} + F_{\alpha+6, \beta+6, \gamma} \\ & + \Pi t_i^6 [F_{\alpha, \beta-6, \gamma-6} + F_{\alpha-6, \beta, \gamma-6} + F_{\alpha-6, \beta-6, \gamma}], \end{aligned} \quad (b)$$

$$\sum t_i^6 t_k^6 t_l^6 \cdot F_{\alpha, \beta, \gamma} = F_{\alpha+6, \beta+6, \gamma+6} + \Pi t_i^6 [F_{\alpha-6, \beta, \gamma} + F_{\alpha, \beta-6, \gamma} + F_{\alpha, \beta, \gamma-6}], \quad (c)$$

Therefore,

$$\sum t_i^6 \cdot F_{\alpha, \beta, \gamma} - \sum t_i^6 t_k^6 \cdot F_{\alpha-6, \beta, \gamma} + \sum t_i^6 t_k^6 t_l^6 \cdot F_{\alpha-12, \beta, \gamma} - \Pi t_i^6 \cdot F_{\alpha-18, \beta, \gamma} = F_{\alpha+6, \beta, \gamma}, \quad (d)$$

$$\sum t_i^6 \cdot F_{\alpha, \beta, \gamma} - \sum t_i^6 t_k^6 \cdot F_{\alpha, \beta-6, \gamma} + \sum t_i^6 t_k^6 t_l^6 \cdot F_{\alpha, \beta-12, \gamma} - \Pi t_i^6 \cdot F_{\alpha, \beta-18, \gamma} = F_{\alpha, \beta+6, \gamma}, \quad (e)$$

$$\sum t_i^6 \cdot F_{\alpha, \beta, \gamma} - \sum t_i^6 t_k^6 \cdot F_{\alpha, \beta, \gamma-6} + \sum t_i^6 t_k^6 t_l^6 \cdot F_{\alpha, \beta, \gamma-12} - \Pi t_i^6 \cdot F_{\alpha, \beta, \gamma-18} = F_{\alpha, \beta, \gamma+6}. \quad (f)$$

$$F_{\alpha, \beta, \gamma} = F_{\beta, \gamma, \alpha} = F_{\gamma, \alpha, \beta}. \quad (g)$$

Making $\alpha = 0$ in the equations (a), (b), (c), (d), (e), (f) and (g), we derive similar equations for $F_{\beta, \gamma}$.

Assuming $\alpha \equiv 0 \pmod{6}$, the discussion is divided into two cases, according as

$$\beta \equiv \gamma \equiv 0 \pmod{2} \quad \text{or} \quad \beta \equiv \gamma \equiv 1 \pmod{2}.$$

Case I.

$F_{\alpha, \beta}$ and $F_{\alpha, \beta, \gamma}$ are symmetric with regard to t_i^6 ; and each is accordingly a rational integral function* of the elementary symmetric functions of t_i^6 .

Case II.

We shall consider

a. Two-lettered forms.

Making $\alpha = 0$ in equation (e), we can express every $F_{\beta, \gamma}$ as an integral function of $F_{\beta', \gamma}$ and $F_{\beta+3, 3, 3}$, where $\beta' \leq 15$. Since $F_{\beta, \gamma} = -F_{\gamma, \beta}$, we have

* Chrystal's Algebra, Vol. I, p. 440.

an expression for every $F_{\beta, \gamma}$ in terms of $F_{\beta', \gamma'}$ and $F_{\beta+3, 3, 3}, F_{\gamma+3, 3, 3}$, where $\beta', \gamma' \leq 15$. Whence β', γ' can have the following sets of values:

$$3, 9; \quad 3, 15; \quad 9, 15.$$

But by (b), $F_{9, 15}$ can be expressed as an integral function of $F_{3, 9}$ and certain $F_{\alpha, \beta, \gamma}$, which shall be discussed under b.

b. Three-lettered forms.

By successive applications of the recursion formulæ (d), (e), (f) and formula (a), we can express any $F_{\alpha, \beta, \gamma}$ as an integral function of $F_{9, 3}, F_{15, 3}$, the elementary symmetric functions of t_i^6 and $F_{\alpha', \beta', \gamma'}$, where the simultaneous values of α', β', γ' are given by

$$\begin{aligned} \alpha' &= 6; \quad 6; \quad 6; \quad 6; \quad 6; \quad 6; \\ \beta' &= 3; \quad 3; \quad 3; \quad 15; \quad 15; \quad 15; \\ \gamma' &= 3; \quad 9; \quad 15; \quad 3; \quad 9; \quad 15. \end{aligned}$$

We shall now show that every $F_{\alpha', \beta', \gamma'}$ can be expressed as an integral function of the elementary symmetric function of t_i^6 , $F_{9, 3}, F_{15, 3}, F_{6, 3, 3}$, and $F_{6, 9, 3}$, and that none of the functions just named can be expressed as integral functions of forms of H of lower degree.

To express rationally and integrally one of these forms in terms of forms of lower degree, it must be involved in an equation with them; its coefficient must be independent of t_i , and, since the equation must be homogeneous, the coefficients of the forms of lower degree are functions of t_i . The method is sufficiently well illustrated by solving one case—that in which the degree of the form is 30.

We have the following set of equations:

$$\begin{aligned} (\Sigma t_i^6) F_{6, 9, 9} &= F_{6, 15, 9} + F_{12, 9, 9} + F_{6, 9, 15}, \\ (\Sigma t_i^6 t_k^6) F_{6, 3, 3} - \Pi t_i^3 (F_{6, 9, 3} - F_{6, 3, 9}) &= F_{12, 9, 9}, \\ (\Sigma t_i^6 t_k^6 t_e^6) F_{9, 3} - \Pi t_i^3 F_{12, 3, 3} &= F_{6, 15, 9} - F_{6, 9, 15}, \end{aligned}$$

which enable us to solve for all forms of degree 30.

Accordingly the elementary symmetric functions of t_i^6 , $F_{9, 3}, F_{15, 3}, F_{6, 3, 3}$, and $F_{6, 9, 3}$ constitute the basis of the System H .

6.—The Invariant Forms as Rational Functions of g_2, g_3 and Δ .

Each invariant function $F_{\alpha, \beta, \gamma}$ is a rational* function of J , where $J = \frac{g_2^3}{\Delta}$, and, therefore, a rational function of g_3 and Δ , where g_3 has the usual signification.†

By equations (α , p. 12), and the relations found in Klein-Fricke, Vol. II, p. 30, we obtain

$$\begin{aligned}\Delta^3 t_1^3 &= \sigma_{\frac{1}{2}, \frac{5}{2}}^3 = \frac{\xi_3 - \xi_4}{-i\sqrt[3]{\Delta}}, \\ \Delta^3 t_2^3 &= \sigma_{\frac{1}{2}, \frac{1}{2}}^3 = \frac{\rho^2 \xi_3 - \xi_4}{-i\sqrt[3]{\Delta}}, \\ \Delta^3 t_3^3 &= \sigma_{\frac{1}{2}, \frac{3}{2}}^3 = \frac{\rho \xi_3 - \xi_4}{-i\sqrt[3]{\Delta}}, \\ \Delta^3 t_4^3 &= \sigma_{\frac{3}{2}, \frac{1}{2}}^3 = -\frac{\xi_4 \sqrt[3]{3}}{\sqrt[3]{\Delta}},\end{aligned}$$

where $\rho = e^{\frac{2\pi i}{3}}$.

‡ Whence

$$\left. \begin{aligned}\Sigma t_i^6 &= 0, \\ \Sigma t_i^6 t_k^6 &= \frac{18}{\Delta^{13}}, \\ \Sigma t_i^6 t_k^6 t_e^6 &= -\frac{216g_3}{\Delta^{20}}, \\ t_1^3 t_2^3 t_3^3 t_4^3 &= \frac{3\sqrt[3]{3}}{i\Delta^{13}}\end{aligned}\right\} \quad (20)$$

and

$$\begin{aligned}F_{9, 3} &= -\frac{36}{i\Delta^{13}}, \\ F_{15, 3} &= -\frac{3 \cdot 216 \sqrt[3]{3} g_3}{\Delta^{20}}, \\ F_{6, 3, 3} &= -\frac{36}{i\Delta^{13}}, \\ F_{6, 9, 3} &= -\frac{3\rho 216 g_3}{\Delta^{20}}.\end{aligned}$$

* Weber, Elliptische Functionen, p. 149.

† Klein-Fricke, Vol. I, p. 15.

‡ Klein-Fricke, Vol. I, p. 630.

From equations (20), it is evident that $t_1^6, t_2^6, t_3^6, t_4^6$ are the roots of the equation

$$y^4 + \frac{18}{\Delta^{13}} y^2 + \frac{216g_3}{\Delta^{20}} y - \frac{27}{\Delta^{26}} = 0.$$

Case II. $n \equiv 0 \pmod{2}$.

7.—The Functions $X_a(u|\omega_1, \omega_2)$. The Functions z_a for Case $n \equiv 0 \pmod{2}$.

The arbitrary constant C_a , in the definition of $X_a(u)$, may be determined so that we may write,* in case n is even,

$$X_a(u|\omega_1, \omega_2) = e^{-\frac{\pi i a}{n} + \frac{3\pi i}{4}} \sqrt{\Delta} \cdot \Delta e^{-G_1 u^2} \sigma_{\frac{a}{n} + \frac{1}{4}, \frac{1}{4}} \left(u|\omega_1, \frac{\omega_2}{n} \right)$$

or,

$$X_a(u|\omega_1, \omega_2) = \sqrt{\frac{2n\pi}{\omega_2}} \sqrt[3]{\Delta} e^{\frac{n\eta_2 u^2}{2\omega_2}} \cdot z^{-a} r^{\frac{a}{2n}} \theta_3 \left(\frac{nn - a\omega_1}{\omega_2} \pi, r^n \right).$$

There are in this case also n linearly independent functions $X_a(u)$ and no more.

Professor Klein has shown† that the linear substitutions on $X_a(u)$ corresponding to the modular substitutions S and T (defined in equation (9)), to be

$$\left. \begin{aligned} S: X'_a(u) &= \varepsilon^{\frac{n}{8} + \frac{a^2}{2}} X_a(u|\omega_1, \omega_2), \\ T: X'_a(u) &= \frac{-1}{\sqrt{n}} \sum_{\beta=0}^{n-1} \varepsilon^{-a\beta} X_\beta(u|\omega_1, \omega_2). \end{aligned} \right\} \quad (21)$$

If we put

$$z_a = X_a(0),$$

there are in the case $n = 4$ three distinct z_a , viz. z_0, z_1, z_2 . Putting $n = 4$ and $u = 0$ in equation (21) and writing $\varepsilon = e^{\frac{2\pi i}{8}}$, we obtain

$$\left. \begin{aligned} S: \quad z'_0 &= \varepsilon z_0, \\ z'_1 &= \varepsilon^2 z_1, \\ z'_2 &= \varepsilon^5 z_2, \\ T: \quad z'_1 &= -\frac{1}{2}(z_0 + 2z_1 + z_2), \\ z'_2 &= -\frac{1}{2}(z_0 - z_2), \\ z'_3 &= -\frac{1}{2}(z_0 - 2z_1 + z_2). \end{aligned} \right\} \quad (22)$$

* Klein-Fricke, Vol. II, p. 285.

† Ib.

We shall designate these substitutions on z_a as S and T . By composition of S and T , we obtain a group of linear substitutions on three variables of order $4\phi(8)\psi(8) = 192$. We shall call this group $\bar{G}_{192}$.

If we regard z_0, z_1, z_2 as homogeneous coordinates in a plane, we may show exactly as in Case I, that *no straight line is unchanged under the substitutions S and T , and that*

$$z_1(z_0 - z_2)(z_0 + z_2) = 0$$

is a triangle invariant under $\bar{G}_{192}$, and the only one.

Put

$$u_0 = z_0 + z_2,$$

$$u_1 = z_0 - z_2,$$

$$u_2 = 2z_1.$$

Then, from equations (22), we derive the substitution group of the u_a . It is given by

$$\left. \begin{array}{l} S: \begin{cases} u'_0 = \varepsilon u_1, \\ u'_1 = \varepsilon u_0, \\ u'_2 = \varepsilon^2 u_2, \end{cases} \\ T: \begin{cases} u'_0 = -u_0, \\ u'_1 = -u_2, \\ u'_2 = -u_1. \end{cases} \end{array} \right\} \quad (23)$$

We shall call the group generated by equations (23) G_{192} .

By a consideration of the theory of transformation of the θ -functions, we may find the relation between $u_{\frac{\alpha}{4}}$ and $\sigma_{\frac{\lambda}{2}, \frac{\mu}{2}}$. They are

$$\left. \begin{array}{l} \sigma_{\frac{1}{4}, \frac{1}{4}} = \frac{(1+i)}{2\sqrt{2}} \frac{(u_{\frac{1}{4}})}{\sqrt[4]{\Delta}}, \\ \sigma_{\frac{1}{4}, \frac{3}{4}} = \frac{(1+i)}{2\sqrt{2}} \frac{\varepsilon u_{\frac{1}{4}}}{\sqrt[4]{\Delta}}, \\ \sigma_{\frac{3}{4}, \frac{1}{4}} = \frac{(1+i)}{i2\sqrt{2}} \frac{\varepsilon u_{\frac{3}{4}}}{\sqrt[4]{\Delta}}. \end{array} \right\} \quad (24)$$

8.—The Invariant Functions of u_a .

We shall now seek the functions of u_a which are invariant under the substitutions of G_{192} .

Reasoning, as in Case I, we can show that every invariant form is an integral function of two-lettered invariant forms, one-lettered invariant forms, and $(u_0 u_1 u_2)^\alpha$, where α is a positive integer. Choosing as our originating term $u_0^\alpha u_1^\beta$ and forming the simplest symmetric function of terms arising from subjecting it to the operators of G_{192} , we may show by reasoning, as in Case I, that every two-lettered invariant form may be obtained by giving to α, β all values in $F_{\alpha\beta}$, where

$$\begin{aligned}\alpha &\equiv 0 \pmod{4}, \\ \beta &\equiv 0 \pmod{4},\end{aligned}$$

and where

$$F_{\alpha\beta} = u_0^\alpha u_1^\beta + \varepsilon^{\alpha+\beta} u_1^\alpha u_0^\beta + u_0^\alpha u_2^\beta + \varepsilon^{\alpha+\beta} u_2^\alpha u_0^\beta + \varepsilon^\alpha u_1^\alpha u_2^\beta + \varepsilon^\alpha u_2^\alpha u_1^\beta. \quad (25)$$

Making $\beta = 0$ in (25), we get all invariant one-lettered functions, and if we write $F_{\alpha,0} = F_\alpha$ we get

$$F_\alpha = 2(u_0^\alpha + \varepsilon^\alpha u_1^\alpha + \varepsilon^\alpha u_2^\alpha). \quad (26)$$

We shall now seek the basis of this system of forms obtained by giving α, β all values in (25) and (26).

By calculation it is shown that

$$F_{4,4} F_{\alpha,\beta} - F_{\alpha,\beta+4} F_4 = F_{\alpha,\beta+8} + (u_0 u_1 u_2)^4 (F_{\alpha,\beta-4}). \quad (a)$$

This is a recursion formula for β , and since

$$F_{\alpha,\beta} = \varepsilon^{\alpha+\beta} F_{\beta,\alpha},$$

it is also a recursion formula for α .

By the use of (a), we can express every $F_{\alpha,\beta}$ as an integral function of $F_{\alpha',\beta'}$, where $\alpha', \beta' \leq 8$. We have left, then, the form $F_{\alpha',\beta'}$, where α', β' may have the pairs of values 0, 4; 0, 8; 4, 4; 4, 8; 8, 8. But

$$\begin{aligned}2F_4^2 &= F_8 - 2F_{4,4}, \\ F_4 F_{4,4} &= 4F_{8,4} - 2\Pi u_i^4, \\ F_{4,4}^2 &= 2F_{8,8} + \Pi u_i^4 F_4.\end{aligned}$$

The basis of the system is, therefore, $F_4, F_{4,4}$ and $(u_0 u_1 u_2)^2$.

Since these forms are invariant under S and T , each of these (and, therefore, every invariant function) can be expressed as a rational function of g_2, g_3 and Δ . We shall now find $F_4, F_{4,4}, (u_0 u_1 u_2)^2$ in terms of these invariants.

* Since $\sigma_{1,1}^4 + \sigma_{0,1}^4 + \sigma_{1,0}^4 = 0$,

$$F_4 = 0.$$

† Now,

$$\begin{aligned}\sigma_{0,1}^4 &= -8\lambda_4, \\ \sigma_{1,0}^4 &= 8\lambda_3, \\ \sigma_{1,1}^4 &= -8(\lambda_3 - \lambda_4).\end{aligned}$$

Therefore, from equations (24),

$$\begin{aligned}u_0^4 &= 2^4 \Delta 8(\lambda_3 - \lambda_4), \\ u_1^4 &= 2^4 \Delta 8\lambda_3, \\ u_2^4 &= -2^4 \Delta 8\lambda_4.\end{aligned}$$

‡ Therefore,

$$\begin{aligned}F_{4,4} &= 2^{14} \Delta g_2, \\ u_0^4 u_1^4 u_2^4 &= 2^{22} \Delta^2.\end{aligned}$$

Hence u_0^4, u_1^4, u_2^4 satisfies the equation

$$x^3 + 2^{14} \Delta g_2 x - 2^{22} \Delta^2 = 0.$$

Now, $u_0^2 u_1^2 u_2^2 = \left(\frac{2\pi}{\omega_2}\right)^3 \frac{\Delta}{\Delta^4} \cdot r^4$, all multiplied into a power series of integral powers of r with real coefficients $= \left(\frac{2\pi}{\omega_2}\right)^3 \Delta$ into a power series of integral powers of r with real coefficients.

If we now approach infinity along positive y -axis, we see that $u_0^2 u_1^2 u_2^2$ has the same sign as Δ . Therefore, we may write

$$u_0^2 u_1^2 u_2^2 = 2^{11} \Delta.$$

9.—Linear Connection between $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ and $X_{\frac{a}{n^2}}(u)$.

We shall show now that $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ is a linear homogeneous function of $X_{\frac{a}{n^2}}(u)$ for any finite value of n .

$$\begin{aligned}\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu) &= F(\omega, \eta) e^{\left[\left(\lambda + \frac{n(n-1)}{2}\right) \eta_1 + \left(\mu + \frac{n(n-1)}{2}\right) \eta_2\right] \left(u - \frac{(\lambda + (n-1)n)\omega_1 + (\mu + (n-1)n)\omega_2}{2}\right)} \\ &\quad \cdot \prod_{m_1, m_2} \sigma \left(u - \frac{\lambda\omega_1 + \mu\omega_2}{n^2} - \frac{m_1\omega_1 + m_2\omega_2}{n}\right), \quad (m_1, m_2 = 0 \dots n-1).\end{aligned}$$

* Klein-Fricke, Vol. II, p. 29.

† Ib., Vol. I, p. 628.

‡ Klein-Fricke, Vol. I, p. 629.

This is a σ -product of n^2 factors multiplied by e with an exponent in which the coefficient of $u = \left[\lambda + \frac{n(n-1)}{2} \right] \eta_1 + \left[\mu + \frac{n(n-1)}{2} \right] \eta_2$, and whose residual sum $s = \lambda \omega_1 + \mu \omega_2 + \frac{n(n-1)}{2} \omega_1 + \frac{n(n-1)}{2} \omega_2$; moreover, there are n^2 different quantities $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$.

If we define n^2 functions by the equations

$$x_i = c_i \prod_{k=0}^{n^2-1} \sigma(u - a_{i,k}). \quad (i = 0 \dots n^2 - 1),$$

where c_i is independent of u and the residual sum,

$$s = \sum_{k=0}^{n^2-1} a_{i,k} = 0, \quad (\text{a})$$

then by *Hermite's Law $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ can be expressed as a linear homogeneous function of x_i .

Also, if n is odd,

$$X_{\frac{\alpha}{n^2}}(u) = f(\omega, \eta) e^{\left(\alpha\eta + \left(\frac{n^2-1}{2}\right)\eta_2\right)\left(u - \alpha\omega_1 + \frac{n^2-1}{2}\omega_2\right)} \cdot \prod_{\mu=0}^{n^2-1} \sigma\left(u - \frac{\alpha\omega_1 + \mu\omega_2}{n^2}\right).$$

From this it appears that $X_{\frac{\alpha}{n^2}}(u)$ is a σ -product of n^2 factors multiplied by e with an exponent which has the coefficient of $u = \alpha\eta_1 + \frac{(n^2-1)}{2}\eta_2$ and whose residual sum

$$s = \alpha\omega_1 + \frac{n^2-1}{2}\omega_2.$$

There are n^2 different quantities $X_{\frac{\alpha}{n^2}}(u)$, and hence, according to Hermite's Law, $X_{\frac{\alpha}{n^2}}(u)$ can be expressed as a linear homogeneous function of x_i defined by equa-

* Klein, Elliptischen Normal Curven der n ter Ordnung, p. 354.

tion (a), and hence $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ can in general be expressed as a linear homogeneous function of $X_{\frac{\alpha}{n^2}}(u)$.

If n is even,

$$X_{\frac{\alpha}{n^2}}(u) = \phi(\omega, \eta) e^{\left(n^2 \left(\frac{\alpha}{n^2} + \frac{1}{2}\right) \eta + \frac{n^2}{2} \eta^2\right) \left(u - \frac{n^2 \left(\frac{\alpha}{n^2} + \frac{1}{2}\right) \omega_1 + \frac{n^2}{2} \omega_2}{2}\right)} \cdot \prod_{\mu} \sigma \left(u - \left[\left(\frac{\alpha}{n^2} + \frac{1}{2}\right) \omega_1 + \frac{\mu + \frac{1}{2}}{n^2} \omega_2\right]\right).$$

Reproducing the argument used above, it follows that, in this case also, $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ can be expressed as a linear homogeneous function of $X_{\frac{\alpha}{n^2}}(u)$, and hence for any finite value of n we may write

$$\sigma_{\frac{0}{n}, \frac{1}{n}}(nu) = \sum_{\alpha=1}^{n^2-1} A_{\alpha} X_{\frac{\alpha}{n^2}}(u), \quad (27)$$

where A_{α} is independent of u .

We shall now determine A_{α} of equation (27) in the case n is even. The theory of the transformation of the theta-functions gives

$$\sigma_{\frac{n}{2}, \frac{n}{2} - \frac{\alpha}{n}}(nu) = \frac{-\frac{1+i}{\sqrt{2}}}{n\sqrt{\Delta}} \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} X_{\frac{n\beta}{n^2}}(u). \quad (28)$$

By equation (7) (Klein-Fricke, Vol. II, p. 27), the $\sigma_{\frac{n}{2}, \frac{n}{2} - \frac{\alpha}{n}}(nu)$ is transformed into another $\sigma_{\frac{\lambda'}{n}, \frac{\mu'}{n}}(nu)$ by both S^k and T^l . We shall indicate this transformation by marking the transformed quantities with a prime.

If we operate on the left side of equation (28) with T , we obtain

$$T: X'_{\frac{n\beta}{n^2}}(u) = \sum_{\gamma=0}^{n^2-1} \varepsilon'^{-n\beta\gamma} X_{\frac{\gamma}{n^2}}(u),$$

where

$$\varepsilon' = e^{\frac{2\pi i}{n^2}}.$$

Whence,

$$\begin{aligned}
 T: \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} X_{\frac{\beta}{n^2}}' &= \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} \sum_{\gamma=0}^{n^2-1} \varepsilon^{l-n\beta\gamma} X_{\frac{\gamma}{n^2}}(u) \\
 &= X_{\frac{0}{n^2}}(u) \left(\sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} \right) + \dots + X_{\frac{r}{n^2}}(u) \left(\sum_{\beta=0}^{n-1} \varepsilon^{-\beta(r+\alpha)} \right) \\
 &\quad + \dots + X_{\frac{n^2-1}{n^2}}(u) \left(\sum_{\beta=0}^{n-1} \varepsilon^{-\beta(n^2-1+\alpha)} \right). \quad (29)
 \end{aligned}$$

It is evident that

$$\sum_{\beta=0}^{n-1} \varepsilon^{-\beta(r+\alpha)} = 0$$

if* $[(r+\alpha), n] = 1$; for, if f is the least number such that

$$\varepsilon^{-(r+\alpha)f} = 1 = \varepsilon^{-fn},$$

then, $f = \frac{sn}{r+\alpha}$. Hence, since $f \leq n$, $s = r + \alpha$. Therefore, $f = n$.

Hence $\varepsilon^{-(r+\alpha)}$ appertains† to the exponent n , and hence $\varepsilon^{-\alpha(r+\alpha)}$ is an n^{th} root of unity, and if $\alpha \neq \alpha'$, $\varepsilon^{-\alpha(r+\alpha)} \neq \varepsilon^{-\alpha'(r+\alpha)}$, where $(\alpha, \alpha' = 0 \dots n-1)$. Moreover, if

$$[(r+\alpha), n] = d, \quad d < n,$$

$\varepsilon^{-(r+\alpha)}$ is an n^{th} root of unity which appertains to the exponent $\frac{n}{d}$. Hence there are $\frac{n}{d}$ different quantities $\varepsilon^{-k(r+\alpha)}$, $(k = 0 \dots (\frac{n}{d} - 1))$. Therefore,

$$\sum_{k=0}^{\frac{n}{d}-1} \varepsilon^{-k(r+\alpha)} = 0. \quad (a)$$

Since

$$\varepsilon^{-k(r+\alpha)} = \varepsilon^{-l(r+\alpha)},$$

if, and only if, $k \equiv l \pmod{\frac{n}{d}}$, and since there are just d quantities $< n$ satisfying this congruence, the quantities $\varepsilon^{-\beta(r+\alpha)}$ may be arranged in d sets, each of

* This notation has the ordinary signification that the G. C. D. of $r+\alpha$, and n is 1.

† Mathews, Theory of Numbers, Part I, p. 185.

which is defined by the summation (a). Therefore,

$$\sum \varepsilon^{-\beta(r+\alpha)} = 0 \text{ if } [(r+\alpha), n] = d, \quad d < n.$$

If $r + \alpha \equiv 0 \pmod{n}$, then $\varepsilon^{-\beta(r+\alpha)} = 1$ for all values of β .

We have now shown that the coefficients of $X_{\frac{\gamma}{n^2}}$ in equation (29) vanishes except when $r + \alpha \equiv 0 \pmod{n}$. Hence we may write

$$T: \sigma'_{\frac{n}{2}, \frac{n-\alpha}{n}}(nu) = \frac{1+i}{i\sqrt[4]{2}\Delta} \sum_{\gamma} X_{\frac{\gamma}{n^2}}(u). \quad (30)$$

where γ satisfies the congruence

$$\gamma + \alpha \equiv 0 \pmod{n}. \quad (b)$$

Now there are just n quantities less than n^2 satisfying this congruence, and since α is less than n , and, therefore, not divisible by n , neither is γ . Hence, in the summation on the left of equation (29), there are n quantities, $X_{\frac{\gamma}{n^2}}(u)$, not one of which is found in summation (28). Giving α all values from 0 to $n-1$, we get n equations, each containing n quantities, $X_{\frac{\gamma}{n^2}}(u)$. Moreover, an $X_{\frac{\gamma}{n^2}}(u)$ occurring in one equation derived from (29), by giving α all values, occurs in no other, for if so,

$$X_{\frac{\gamma}{n^2}}(u) = X_{\frac{\gamma'}{n^2}}(u).$$

Whence $\gamma + \alpha \equiv \gamma' + \alpha \pmod{n^2}$. That is, $\gamma - \gamma' + \alpha - \alpha' \equiv 0 \pmod{n}$, which is impossible. We now apply S to one of the equations obtained by giving α all values in (30):

$$S^k: \sigma'_{\frac{n}{2}, \frac{n-\alpha}{n}}(nu) = \frac{1+i}{i^{k+1}\sqrt[4]{2}\Delta} \sum \varepsilon^{(\frac{n^2}{8} + \frac{\gamma^2}{2})^k} X_{\frac{\gamma}{n^2}}(u). \quad (31)$$

Let us now enquire for a value of k such that by an application of S^k to the summation on the left side of one of the equations (29), produces the same summation multiplied by a constant.

The evident conditions are that any $X_{\frac{\gamma}{n^2}}(u)$ that occurs in the given summation also occurs in the resulting summations, and that any two quantities,

$X_{\frac{\gamma}{n^2}}(u)$, $X_{\frac{\gamma}{n^2}}(u)$, $X_{\frac{\gamma}{n^2}}(u)$ shall have the same coefficient. The last condition is equivalent to

$$\varepsilon' \left(\frac{n^2}{8} + \frac{\gamma^2}{2} \right)^k = \varepsilon' \left(\frac{n^2}{8} + \frac{\gamma'^2}{2} \right)^k.$$

We may now show that this is equivalent to

$$k \equiv 0 \pmod{n}.$$

Hence, giving k in equation (31) all values, we can derive n , and only n , different equations.

Hence, the values of A_α in equation (27) may be obtained by giving to α all values from 0 to $n - 1$, in equation (29), and to k all values from 0 to $n - 1$ in equation (31). We shall, at the same time, arrange these n^2 equations in n sets, each set containing n equations, and in every equation of each set, the left side is a linear homogeneous expression of the same n quantities, $X_{\frac{\gamma}{n^2}}(u)$, while some $\sigma_{\frac{\lambda}{n}, \frac{\mu}{n}}(nu)$ constitutes the right side.

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